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Visible lattice points in the sphere ☆

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Abstract

The number of visible (primitive) lattice points in the sphere of radius R is well approximated by $\frac{4\pi}{3\zeta(3)}R^3$. We consider an integral expression involving the error term $E^*(R)$, which leads to $E^*(R)=\Omega(R(\log R)^{1/2})$. This is comparable to what is known in the sphere problem. We can avoid the use of the second power moment (which is in this case unknown) by employing an auxiliary trigonometric series correlated to $E^*(R)$. This approach to prove Ω-results seems to be new and could be useful in other problems.

MSC: 11P21; 11N37; 11K38

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1. Introduction and notation

Let $E^*(R)$ be the error term in the approximation of the number of visible lattice points in the sphere of radius R,

$$E^*(R) = \#\left\{\vec{n} \in \mathbb{Z}^3 \colon \|\vec{n}\|_2 \leqslant R,\ \gcd(n_1,n_2,n_3)=1\right\} - \frac{4\pi}{3\zeta(3)}R^3.$$

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It is easy to prove that it is possible to translate the best known exponent for the sphere problem into $E^*(R)$; in particular from [5] and Lemma 2.1 one deduces at once that $E^*(R) = O(R^{21/16+\epsilon})$ for every $\epsilon > 0$. This is in high contrast with the two-dimensional case in which nontrivial exponents are proved only after assuming Riemann Hypothesis (see [11] for the best result of this kind). On the other hand, in the three-dimensional case it is harder to study the oscillations of the error term using harmonic analysis than in the two-dimensional case even disregarding visibility condition. This is due to the slower decay of Fourier transforms (compare Theorem 13.5 of [6] and [10]). Recently W.G. Nowak [8] has proved an Ω-result for visible points in the circle. In this paper we intend to deal with, by different techniques, the three-dimensional case.

Consider $M = R(\log R)^{-1/3}$ and the function

$$g(t) = \sum_{n \leqslant M^2} \frac{\cos(2\pi t\sqrt{n})}{\sqrt{n}}.$$

We are interested in the integral

$$I(R) := \int g(t)E^*(t)\,d\nu(t)$$

where $d\nu$ is a probability measure

$$d\nu(x) = R^{-1}\psi(x/R)\,dx$$

with $\psi \in C_0^\infty((1,2))$ and $\int \psi = 1$.

Our result reads

Theorem 1.1. *For $R > 1$*

$$I(R) = -\frac{14}{\pi^2} C R \log R + O\big(R(\log R)^{5/6}\big) \tag{1}$$

where $C = \int x\psi(x)\,dx$.

This reveals a correlation between the functions $g(t)$ and $E^*(t)$. In particular one deduces (using Lemma 2.5)

$$\int \big|E^*(t)\big|^2\,d\nu(t) \gg \frac{R^2(\log R)^2}{\int |g(t)|^2\,d\nu(t)} \gg R^2 \log R$$

which leads to

Corollary 1.2. *For $R > 1$*

$$E^*(R) = \Omega\big(R(\log R)^{1/2}\big). \tag{2}$$

This is comparable to what is known in the case of the sphere problem [9] (see [10] for the two-sided Ω-result).

In the proof, as usual, $r_3(n)$ will denote the number of representations of n as a sum of three squares, and we shall write

$$a_n = \frac{r_3(n)}{\sqrt{n}}\hat{\phi}(\sqrt{n}/M)$$

where $\phi \in C_0^\infty((-1,1))$ is an arbitrary positive even function such that $\int \phi = 1$. Hence a_n is similar to a constant on average for $n \in [1, M^2]$ and a_n is small for n much larger than M^2.

Throughout the paper, the implied O-constants may depend on the choice of ψ and ϕ. We shall employ the notation $x \asymp X$ meaning $X \leqslant x < 2X$ or in general $X \ll x \ll X$.

2. Auxiliary lemmata

Let $E(R)$ be the error term for the sphere problem,

$$E(R) = \sum_{1\leqslant n\leqslant R^2} r_3(n) - \frac{4\pi}{3}R^3.$$

The relation between $E(R)$ and $E^*(R)$ is straightforward:

Lemma 2.1. *For $R > t > 1$*

$$E^*(t) = \sum_{d\leqslant R} \mu(d)E(t/d) + o(t).$$

Proof. By the Möbius inversion formula the number of representations of n as a sum of three coprime squares is $\sum_{d^2|n} \mu(d)r_3(n/d^2)$, then the number of visible points in the sphere is

$$\sum_{n\leqslant t^2}\sum_{d^2|n} \mu(d)r_3\bigl(n/d^2\bigr) = \sum_{d\leqslant t}\mu(d)\sum_{n\leqslant t^2/d^2} r_3(n).$$

The inner sum can be substituted by $4\pi t^3 d^{-3}/3 + E(t)$, obtaining

$$E^*(t) = \sum_{d\leqslant t} \mu(d)E(t/d) + o(t).$$

The error term follows by applying partial summation and using $\sum_{n\leqslant t} \mu(n) = o(t)$. We finish the proof noticing that $E(u) = -(4\pi/3)u^3$ for $0 < u < 1$. □

Remark. This formula holds in every dimension D. Formally, $E(x) = O(x^\alpha)$ gives $E^*(x) = O(x^\alpha)$ for $\alpha > 1$ and $E^*(x) = O(x)$ for $\alpha < 1$. For this reason, for $D = 2$ (visible points in a circle) the Hardy conjecture ($\alpha = 1/2 + \epsilon$, $\forall\epsilon > 0$) would give only a trivial result. For $D = 3$ the conjecture is $\alpha = 1 + \epsilon$ but note that for instance $E(x) = O(x \log x)$ implies that $E^*(x) = O(x \log^2 x)$, that is to say the extra logarithm can ruin average results for $E^*(x)$ if the cancellation induced by Möbius function is disregarded. In connection with this, we need to utilize precise approximations of $E(t)$, because in this case an error term like $O((\log t)^{1/2})$ absorbs the Ω order in (2).

Lemma 2.2. *For* $R > 1/2$

$$E(R) = -\frac{R}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}\cos(2\pi R\sqrt{n}) + T(R) + U(R)$$

where

$$T(R) = \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{a_n}{n}\sin(2\pi R\sqrt{n}) \quad \textit{and} \quad U(R) \ll 1 + \sum_{k=1}^{\infty} r_3(k)\chi_k(R)$$

with χ_k *the characteristic function of the interval* $[\sqrt{k} - 1/M, \sqrt{k} + 1/M]$.

Proof. Let $g(x) = x$ if $|x| \leqslant R$ and $g(x) = 0$ otherwise. Choosing f in Lemma 2.1 of [2] as the convolution of g and $M\phi(Mx)$, one gets

$$\sum_{n=1}^{\infty} r_3(n)\frac{f(\sqrt{n})}{\sqrt{n}} = \frac{4\pi}{3}R^3 - \frac{R}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}\cos(2\pi R\sqrt{n}) + T(R) + O(1).$$

The main term is computed either directly or differentiating the Fourier transform, $\hat{f}$, at zero and the sine Fourier transform, $\tilde{f}$, of Lemma 2.1 of [2] is

$$\tilde{f}(\sqrt{n}) = i\hat{g}(\sqrt{n})\hat{\phi}\left(\frac{\sqrt{n}}{M}\right) = \left(\frac{-R\cos(2\pi R\sqrt{n})}{\pi\sqrt{n}} + \frac{\sin(2\pi R\sqrt{n})}{2\pi^2 n}\right)\hat{\phi}\left(\frac{\sqrt{n}}{M}\right).$$

The left-hand side equals $\sum_{n\leqslant R^2} r_3(n)$ if $|R - \sqrt{N}| > M^{-1}$ where N is the nearest integer to R^2. Otherwise one should add the contribution of the $r_3(N)$ (note that for $x > 0$, $f(x)/x$ and the characteristic function of $(0, R]$ only differ in the interval $[R - M^{-1}, R + M^{-1}]$). This gives the bound for $U(R)$. □

Lemma 2.3. *We have*

$$\sum_{n\leqslant N} r_3^2(n) \sim C_1 N^2,$$

with C_1 *a positive constant. Indeed* $C_1 = 8\pi^4/(21\zeta(3))$.

Proof. This statement can be proved using the properties of the Rankin–Selberg convolution or from the application of the circle method to study the asymptotics of $\int |\theta(z)|^6\,dz$ [1,3] (see also [7] for a general theorem). □

Lemma 2.4. *We have*

$$\sum_{n\leqslant N}\mu^2(n)r_3(n)=\frac{28}{3\pi}N^{3/2}+O\big(N^{5/4}(\log N)^2\big).$$

Remark. This formula has been calculated without trying to reduce the error term to the minimum. It is possible that the techniques of [4] could be employed to diminish its exponent to $1+\epsilon$.

Proof. The relation between $r_3(n)$ and the class number (see (1.2) in [2]), implies that

$$\frac{r_3(n)}{\sqrt{n}}\ll\sum_{d^2|n}d^{-1}L(1,\chi_{n/d^2})\quad\text{with }\chi_k(\cdot)=\left(\frac{-4k}{\cdot}\right),$$

whence $L(1,\chi_m)\ll\log m$ leads to $r_3(n)=O(\sqrt{n}\,(\log n)^2)$, and by the elementary identity $\mu^2(n)=\sum_{d^2|n}\mu(d)$

$$\sum_{n\leqslant N}\mu^2(n)r_3(n)=\sum_{d<N^{1/4}}\mu(d)\sum_{\substack{n\leqslant N\\ n\equiv 0\ (\mathrm{mod}\ d^2)}}r_3(n)+O\big(N^{5/4}(\log N)^2\big).$$

The inner sum is

$$\sum_{\substack{1\leqslant r_1,r_2,r_3\leqslant d^2\\ r_1^2+r_2^2+r_3^2\equiv 0\ (\mathrm{mod}\ d^2)}}\ \sum_{\substack{a_1^2+a_2^2+a_3^2\leqslant N\\ a_j\equiv r_j\ (\mathrm{mod}\ d^2)}}1=w\big(d^2\big)\left(\frac{4\pi}{3}N^{3/2}d^{-6}+O\big(Nd^{-4}\big)\right)$$

with $w(d^2)$ being the number of solutions of the equation $r_1^2+r_2^2+r_3^2=0$ in $\mathbb{Z}/d^2\mathbb{Z}$. w is a multiplicative function for which $w(2^2)=2^3$ and $w(p^2)=p^4$ if p is an odd prime. Therefore

$$\begin{aligned}\sum_{n\leqslant N}\mu^2(n)r_3(n)&=\frac{4\pi}{3}N^{3/2}\sum_{d<N^{1/4}}\frac{\mu(d)w(d^2)}{d^6}+O\big(N^{5/4}(\log N)^2\big)\\&=\frac{4\pi}{3}N^{3/2}\prod_p\left(1-\frac{w(p^2)}{p^6}\right)+O\big(N^{5/4}(\log N)^2\big)\end{aligned}$$

and the infinite product equals $(1-2^{-2})^{-1}(1-2^{-3})/\zeta(2)=7/\pi^2$. $\square$

Lemma 2.5. *For g defined as in the introduction*

$$\int\bigg|\sum_{n\leqslant M^2}\frac{e(t\sqrt{n})}{\sqrt{n}}\bigg|^2d\nu(t)\sim 2\log R\quad\textit{and}\quad\int\big|g(t)\big|^2d\nu(t)\sim\log R.$$

Proof. We begin proving the first expression. Expanding the square and changing variables we get

$$\sum_{n_1\leqslant M^2}\sum_{n_2\leqslant M^2}\frac{1}{\sqrt{n_1}\sqrt{n_2}}\int_1^2 e\big(yR(\sqrt{n_2}-\sqrt{n_1})\big)\psi(y)\,dy.$$

For $n_1=n_2$, the diagonal terms, we obtain

$$\sum_{n\leqslant M^2}\frac{1}{n}\int_1^2\psi(y)\,dy\sim 2\log R.$$

Hence, the first expression of the lemma will be proved by checking that the contribution of the other terms is smaller.

Indeed, for $n_1\neq n_2$ we have

$$\sum_{\substack{n_1,n_2\leqslant M^2\\ n_1\neq n_2}}\frac{1}{\sqrt{n_1}\sqrt{n_2}}\hat{\psi}\big(R(\sqrt{n_1}-\sqrt{n_2})\big).$$

In the range of summation $|R(\sqrt{n_1}-\sqrt{n_2})|\gg 1$, by using $\hat{\psi}(R(\sqrt{n_1}-\sqrt{n_2}))\ll|R(\sqrt{n_1}-\sqrt{n_2})|^{-1}$ and $\sqrt{n_1}-\sqrt{n_2}=(n_1-n_2)/(\sqrt{n_1}+\sqrt{n_2})$ we find that the last sum is $o(\log R)$. This completes the proof of the first expression.

Next, writing the second expression in function of exponentials, opening the square and by the above argument, we get that its asymptotics is given by

$$\log R+\sum_{n_1\leqslant M^2}\sum_{n_2\leqslant M^2}\frac{\hat{\psi}(-R(\sqrt{n_1}+\sqrt{n_2}))+\hat{\psi}(R(\sqrt{n_1}+\sqrt{n_2}))}{4\sqrt{n_1}\sqrt{n_2}}.$$

Finally appealing again to the decay of $\hat{\psi}$, it is not difficult to check that the contribution of the last sum is $o(\log R)$. □

3. Proof of the result

We can write

$$I(R)=I_1(R)+I_2(R)+I_3(R)+O(R),$$

where $I_1(R)$, $I_2(R)$ and $I_3(R)$ give the contribution to $I(R)$ of each summand in Lemma 2.2 after substituting in Lemma 2.1. For example

$$I_2(R)=\sum_{d<2R}\mu(d)\int g(t)T(t/d)\,d\nu(t).$$

With these definitions, (1) is a consequence of the following results.

Proposition 3.1. *For $R > 2$*

$$I_1(R) = -\frac{14}{\pi^2} C R \log R + O\big(R(\log R)^{5/6}\big).$$

Proposition 3.2. *For $R > 2$*

$$I_2(R) = O\big((\log R)^4\big).$$

Proposition 3.3. *For $R > 2$*

$$I_3(R) = O\big(R(\log R)^{5/6}\big).$$

The crux of the argument is in the proof of Proposition 3.1. We shall employ the following lemma.

Lemma 3.4. *Given N, D and L positive real numbers such that $1 \leqslant D \leqslant 2\sqrt{N}$ and $\sqrt{N}/DL < 1$, we define for each $n \in \mathbb{N}$ the set*

$$C_n = \big\{D \leqslant d < 2D\colon\ 0 \neq \|n/d^2\| < \sqrt{N}/DL\big\}$$

where $\|\cdot\|$ denotes the distance to the nearest integer, and consider the sum

$$S(N, D, L) = \sum_{N \leqslant n < 2N} a_n |C_n|.$$

Then

$$S(N, D, L) \ll (\log N)^2 \mathcal{A}, \tag{3}$$

$$S(N, D, L) \ll N^{1/2} \mathcal{A}^{1/2} \quad \text{if } D^4 < N < L^2, \tag{4}$$

$$S(N, D, L) \ll \mathcal{A} \quad \text{if } DN^{13/32} > L \tag{5}$$

with $\mathcal{A} = N^{3/2} L^{-1} \min(1, M^{40} N^{-20})$.

Proof. Using $r_3(n) = O(\sqrt{n}\,(\log n)^2)$ and $\hat{\phi}(x) \ll (1 + |x|^{40})^{-1}$,

$$a_n \ll (\log N)^2 \min\big(1, M^{40} N^{-20}\big).$$

Given m and d, there are at most $2d^2\sqrt{N}/DL$ values of n with $0 \neq |n/d^2 - m| < \sqrt{N}/DL$, then

$$\sum_{N \leqslant n < 2N} |C_n| \ll \sum_{m \asymp N/D^2} \sum_{d \asymp D} \frac{d^2\sqrt{N}}{DL} \ll N^{3/2} L^{-1}.$$

Combining this and the previous bound, one obtains (3).

If $0 \neq \|n/d^2\| < \sqrt{N}/DL$ then there exists an $h \in \mathbb{Z}$ such that $|h| < d^2\sqrt{N}/DL$ and $d^2 \mid n-h$. By Lemma 2.3, after Cauchy's inequality, we get

$$S^2(N,D,L) \ll N \min\bigl(1, M^{80}N^{-40}\bigr) \sum_{N \leqslant n < 2N} \Biggl(\sum_{0<|h| \ll D\sqrt{N}/L} \sum_{\substack{D \leqslant d < 2D \\ d^2 \mid n-h}} 1 \Biggr)^2. \tag{6}$$

The sum equals

$$\sum_{\substack{D \leqslant d_1 < 2D \\ D \leqslant d_2 < 2D}} \sum_{\substack{0<|h_1| \ll D\sqrt{N}/L \\ 0<|h_2| \ll D\sqrt{N}/L}} \bigl|\bigl\{N \leqslant n < 2N\colon\ n \equiv h_1 \pmod{d_1^2},\ n \equiv h_2 \pmod{d_2^2}\bigr\}\bigr|.$$

The contribution to the last sums of the "diagonal" terms $h_1 = h_2 = h$ is given by

$$\sum_{0<|h| \ll D\sqrt{N}/L} \sum_{D \leqslant d_1, d_2 < 2D} \frac{N}{[d_1,d_2]^2} \ll \frac{N^{3/2}}{LD^3} \sum_{D \leqslant d_1, d_2 < 2D} (d_1,d_2)^2 \ll N^{3/2}L^{-1}$$

where $[\cdot,\cdot]$ and $(\cdot,\cdot)$ denote the least common multiple and the greatest common divisor, respectively (note that $[d_1,d_2]^2 < N$ because $D^4 < N$).

By the Chinese Remainder Theorem, $n \equiv h_1 \pmod{d_1^2}$, $n \equiv h_2 \pmod{d_2^2}$ has a single solution $(\mathrm{mod}\ [d_1,d_2]^2)$ when $(d_1,d_2)^2 \mid h_1 - h_2$. Thus the contribution to the sum of nondiagonal terms is

$$\sum_{D \leqslant d_1, d_2 < 2D} \biggl(\frac{D\sqrt{N}}{L}\biggr)^2 (d_1,d_2)^{-2} \frac{N}{[d_1,d_2]^2} \ll N^2L^{-2} \leqslant N^{3/2}L^{-1}$$

if $N < L^2$. Substituting these calculations in (6), it follows that

$$S(N,D,L) \ll N^{5/4}L^{-1/2} \min\bigl(1, M^{40}N^{-20}\bigr) \ll N^{1/2}\mathcal{A}^{1/2},$$

which proves (4).

To prove (5) we start as in the proof of (3), noting that having fixed m and d there are at most $2d^2\sqrt{N}/DL$ values of n with $|n - md^2| < d^2\sqrt{N}/DL$; then

$$S(N,D,L) \ll \sum_{m \asymp N/D^2} \sum_{d \asymp D} \sum_{n \in I_{m,d}} a_n$$

where $I_{m,d}$ is an interval contained in $\{x\colon\ |x - md^2| \leqslant 4D\sqrt{N}/L\}$. Applying (1.4) of Theorem 1.1 of [2] with $R^2 = md^2$ and $RH \asymp D\sqrt{N}/L$ one gets (note that $S(R,H) \ll R^2H$ is trivial for $1 \leqslant H \ll R$)

$$\frac{\sum_{n \in I_{m,d}} a_n}{\min(1, M^{40}N^{-20})} \ll D\sqrt{N}/L + (D\sqrt{N}/L)^{2/3}N^{1/32} + 1 \ll D\sqrt{N}/L,$$

which leads to

$$S(N, D, L) \ll N/D^2 \cdot D \cdot D\sqrt{N}/L \cdot \min\bigl(1, M^{40}N^{-20}\bigr) = \mathcal{A}$$

and the proof is finished. □

Proof of Proposition 3.1. Recalling the definition of $I_1(R)$, we have

$$I_1(R) = -\frac{1}{\pi} \sum_{d<2R} \frac{\mu(d)}{d} \sum_{m \leqslant M^2} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{nm}} I(n, m, d)$$

where

$$\begin{aligned} I(n, m, d) &= \int t \cos(2\pi t \sqrt{m}\,) \cos\left(2\pi \frac{t}{d} \sqrt{n}\right) d\nu(t) \\ &= R\hat{\eta}\left(R\left(\sqrt{m} - \frac{\sqrt{n}}{d}\right)\right) + O\bigl(R^{-2}\bigr), \end{aligned}$$

η being the even extension of $x\psi(x)/4$. We shall distinguish two situations: the one where $n = d^2 m$ (diagonal case) and the other where $n \neq d^2 m$ (off-diagonal case).

Diagonal case. By the formula for $I(n, m, d)$,

$$I(n, m, d) = \frac{1}{2} C R + O\bigl(R^{-2}\bigr).$$

Hence its contribution to $I_1(R)$ is

$$-\frac{CR}{2\pi} \sum_{n=1}^{\infty} \sum_{\substack{d^2 | n \\ \sqrt{n}/M < d < 2R}} \mu(d) \frac{a_n}{n} + O(1).$$

Using the formula $\sum_{d^2|n} \mu(d) = \mu^2(n)$, the inner summation is

$$\mu^2(n) + O(\sqrt{n}/M) + O\biggl(\sum_{\substack{d^2 | n \\ d \geqslant 2R}} 1\biggr).$$

By the decay of $\hat{\phi}$ the O terms give a bounded quantity after summing over n

$$\frac{1}{M} \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} + \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{\substack{d^2 | n \\ d \geqslant 2R}} 1 \ll 1 + \sum_{k=1}^{\infty} \sum_{d \geqslant 2R} \frac{a_{kd^2}}{kd^2} \ll 1.$$

Therefore, by Lemma 2.4 and partial summation, the diagonal contribution is

$$-\frac{CR}{2\pi}\sum_{n=1}^{\infty}\mu^2(n)\frac{a_n}{n}+O(R)=-\frac{14}{\pi^2}CR\log M+O(R).$$

Off-diagonal case. We shall consider firstly the case $0\neq|\sqrt{m}-d^{-1}\sqrt{n}|<(4R)^{-1}$, note that this implies $0\neq|m-n/d^2|<\sqrt{n}/dR$ and $d<2\sqrt{n}$. Using the trivial estimate $I(n,m,d)\ll R$, the contribution of these terms to $I_1(R)$ is bounded by

$$R\sum_{d<2R}\frac{1}{d}\sum_{0\neq|m-n/d^2|<\frac{\sqrt{n}}{dR}}\frac{a_n}{\sqrt{nm}}\ll R\sum_{d<2R}\sum_{0\neq|m-n/d^2|<\frac{\sqrt{n}}{dR}}\frac{a_n}{n}.$$

In the range $n\geqslant d^2R^2$ the inner sum is bounded by $\sum_{n\geqslant d^2R^2}a_n/(dR\sqrt{n})$ giving less than $O(R)$. Hence the terms with $0\neq|\sqrt{m}-d^{-1}\sqrt{n}|<(4R)^{-1}$ contribute

$$R\sum_{d<2R}\sum_{0<\|\frac{n}{d^2}\|<\frac{\sqrt{n}}{dR}<1}\frac{a_n}{n}+O(R).$$

In the same way one can consider the terms with $\lambda/8R\leqslant|\sqrt{m}-d^{-1}\sqrt{n}|<\lambda/4R$, $\lambda\geqslant 1$, and by the decay of $\hat{\eta}$ (represented below as λ^{-50}) we can assume that λ is less than a power of R, say $\lambda<R$ which implies again $d<2\sqrt{n}$. Then the whole off-diagonal contribution is controlled by

$$R\sum_{\lambda=2^i<R}\lambda^{-50}\sum_{d<2R}\sum_{0<\|\frac{n}{d^2}\|<\frac{\lambda\sqrt{n}}{dR}<1}\frac{a_n}{n}+O(R),$$

using the decay of a_n if $\lambda<(\log R)^{1/4}$ and the factor λ^{-50} otherwise to omit the terms with $\lambda\sqrt{n}/(dR)\geqslant 1$.

After all of these reductions, with the notation of Lemma 3.4 the off-diagonal contribution is bounded by

$$R\sum_{\lambda=2^i<R}\lambda^{-50}\sum_{N=2^j}N^{-1}\sum_{\lambda R^{-1}\sqrt{N}<D=2^k<2\sqrt{N}}S\bigl(N,D,R\lambda^{-1}\bigr)+O(R).$$

Employing (3) we can see that the sum over $N\geqslant(R\lambda^{-1})^2$ is absorbed by $O(R)$. The inequality

$$R\lambda^{-50}N^{-1}S\bigl(N,D,R\lambda^{-1}\bigr)\ll\lambda^{-49}N^{1/4}R^{1/2}\min\bigl(1,M^{20}N^{-10}\bigr)$$

follows from Lemma 3.4 using (3) for $N<(R\lambda^{-1})^{8/5}$; (4) for $(R\lambda^{-1})^{8/5}\leqslant N\leqslant(R\lambda^{-1})^2$ and $D<(R\lambda^{-1})^{2/5}$ (to assure $D^4<N$); and (5) for $(R\lambda^{-1})^{8/5}\leqslant N\leqslant(R\lambda^{-1})^2$ and $D\geqslant(R\lambda^{-1})^{2/5}$.

Whence the contribution coming from the off-diagonal terms is

$$\sum_{\lambda=2^i<R}\lambda^{-49}\sum_{N=2^j}\sum_{D=2^k<2\sqrt{N}}N^{1/4}R^{1/2}\min\bigl(1,M^{20}N^{-10}\bigr)\ll R(\log R)^{5/6},$$

completing the proof of the result. □

Proof of Proposition 3.2. The diagonal terms contribute

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{3/2}} \sum_{d<2R,\, d^2|n} \mu(d)d \ll \sum_{n=1}^{\infty} \frac{a_n}{n} \ll \log M.$$

The remaining terms give, using trivial bounds,

$$\frac{1}{2\pi^2 R} \sum_{d<2R} \sum_{n\neq md^2} \frac{a_n}{n\sqrt{m}} \cdot \frac{1}{|\sqrt{m}-\sqrt{n}/d|},$$

which is bounded by $O((\log R)^4)$. $\square$

Proof of Proposition 3.3. By the bound for $U(R)$ in Lemma 2.2, Cauchy's inequality and Lemma 2.5 it follows that

$$\begin{aligned} I_3(R) &\ll R + \sum_{d<2R} \int \big|g(t)\big| \sum_{k=1}^{\infty} r_3(k)\chi_k(t/d)\,d\nu(t) \\ &\ll R + (\log R)^{1/2} \sum_{d<2R} \left(\int \left| \sum_{k=1}^{\infty} r_3(k)\chi_k(t/d) \right|^2 d\nu(t) \right)^{1/2}. \end{aligned}$$

Opening the square and changing variables, the integral is bounded by

$$\frac{d}{R} \sum_{j,k} r_3(j) r_3(k) \int\limits_{R/d}^{2R/d} \chi_j(t)\chi_k(t)\,dt.$$

Note that $\chi_j(t)\chi_k(t) = 0$ if $|\sqrt{j}-\sqrt{k}| > 2/M$; therefore the last expression is majorized by

$$\frac{d}{RM} \sum_{\substack{j,k\asymp R^2/d^2 \\ |\sqrt{j}-\sqrt{k}|\leqslant 2/M}} r_3(j)r_3(k) \ll \frac{d}{RM} \sum_{\substack{j,k\asymp R^2/d^2 \\ |j-k|\ll R/dM}} r_3(j)r_3(k).$$

By the elementary inequality $\sum_{l\leqslant L}\sum_n a_n a_{n+l} \leqslant L\|\boldsymbol{a}\|_2^2$ we have

$$I_3(R) \ll R + (\log R)^{1/2} \sum_{d<2R} \left(\frac{d}{RM}\left(\frac{R}{Md}+1\right) \sum_{k\asymp R^2/d^2} r_3^2(k) \right)^{1/2},$$

and Lemma 2.3 gives $I_3(R) \ll R(\log R)^{5/6}$. $\square$

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